General method of synchronization

Liu Zonghua
CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China;
Graduate School, China Academy of Engineering Physics, P.O. Box 8009, Beijing 100088, People's Republic of China;
and Department of Physics, Guangxi University, Nanning, People's Republic of China

Chen Shigang
Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, People's Republic of China
(Received 17 December 1996)

We discuss a general approach for chaotic synchronization of dynamical systems that is based on adjusting the response-system parameter. Numerical simulation shows that this method is robust against external noise. [S1063-651X(97)07306-6]

PACS number(s): 05.45.+b

I. INTRODUCTION

Since 1990 chaos synchronization has been a topic of great interest. Pecora and Carroll [1] considered how identical or almost identical chaotic systems can be synchronized by a chaotic reference signal so that the two systems follow the same chaotic orbit. He and Vaidya [2] showed how this synchronization can be understood in many representative cases by the existence of a global Lyapunov function of the difference signals.

Usually two dynamical systems are called synchronized if the distance between their states converges to zero for $t \rightarrow \infty$. Recently [3,4], a generalization of this concept for unidirectionally coupled systems was proposed, where two systems are called synchronized if a function relation exists between the states of both systems. On the other hand, Ref. [5] points out that when all exponents are negative, except for a few that take a zero value, one may find cases in which the copy of the subsystem reproduces the original despite the fact that the distance between the two subsystems does not converge to zero. A zero conditional Lyapunov exponent would mean that the distance between the drive and response remains constant on average. It allows other possibilities that may be of interest from the scientific and technical points of view, for example, the amplification of the drive attractor and the shift of it to a different region of phase space.

One possible use of the ability to synchronize chaotic systems is in secure communications [6]. A sender of information might add a very large chaotic component to the information-containing signal, thus masking the information in the signal from any third party who intercepts it. Another way [7] of using chaos in communication is to control the dynamics of a chaotic oscillator so that it follows a given sequence in its symbolic dynamics. Since this sequence can be controlled, it can be used to transmit information. Recently, Kocarev and co-workers [8,9] presented a general approach for chaotic synchronization of dynamical systems. The basic idea of the synchronization approach consists in a decomposition of a given (chaotic) system into an active and a passive part, where different copies of the passive part synchronize when driven by the same active component. This approach improves the encoding-decoding schemes and allows us to recover the information signal exactly.

Reference [1] gives a listing of the various subsystems and driving components for the Lorenz and Rössler systems and their sub-Lyapunov exponents; see Table I.

From this table we know that the Rössler system cannot be synchronized when one choses (y,z) or (x,y) as the response system because it has positive sub-Lyapunov exponents; for the Lorenz system, the response system (x,y) cannot be synchronized. If one can change the positive sub-Lyapunov exponents into negative ones, then the synchronization of the response system can be implemented. References [10,11] use a small feedback perturbation to change the positive sub-Lyapunov exponents into negative ones and implement the synchronization.

Reference [12] uses a nonlinear feedback to do the same thing. In this paper, we present two approaches to implement the synchronization of response systems that cannot be synchronized in [1]. One changes the positive sub-Lyapunov exponents to negative by both using the drive variable and adjusting the parameter of the response system, or only by adjusting the parameter of the response system. The adjusting parameter range is given by calculating the sub-Lyapunov spectra. It is necessary to point out that our method is different from Lai and Grebogi's method [13], which is based on the idea of controlling chaos by Ott, Grebogi, and Yorke [14]. Our method applies the Routh-Hurwitz stability criterion to determine the variation of the parameter.

II. METHOD

Consider a genaral dynamical model that displays chaotic behavior:

TABLE I. Subsystems and driving components for the Lorenz and Rössler systems and their sub-Lyapunov exponents.

System	Drive	Response	Sub-Lyapunov exponent
Rössler	x	(y,z)	$(+0.2,-8.89)$
$a=0.2$, $b=0.2$	y	(x,z)	$(-0.056,-8.81)$
$c=9.0$	z	(x,y)	$(+0.1,+0.1)$
Lorenz	x	(y,z)	$(-1.81,-1.86)$
$\sigma=10$, $b=8/3$	y	(x,z)	$(-2.67,-9.99)$
$\gamma=60.0$	z	(x,y)	$(+0.0108,-11.01)$

$$\dot{x}_i = f_i(\mathbf{x}, a_i) \quad i = 1, \ldots, n, \tag{1}$$

where a_i is the parameter of system and $n \geqslant 3$. Equation (1) is now taken as the driving system. One of the system variables x_i may be chosen to be a drive signal, while the remaining variables can be duplicated as a response (driven) system. The global system consists of both the driving and the driven systems, linked by the drive signal. If we use x_j as the drive variable, the response system consists of $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$. Generally speaking, the eigenvalues of the Jacobian matrix of the response system cannot all be negative, so the behavior of the response system cannot always be synchronized with that of the driving system by Pecora and Carroll's method [1]. To solve this problem we adjust the parameter a_i of the response system as

$$a_i = \overline{a_i} - \epsilon_i (x_i - x_{iap}), \tag{2}$$

where $\overline{a_i}$ is the nominal value of a_i, ϵ_i is the control coefficient, x_i is the orbit of the response system, and x_{iap} is the orbit of the driving system. The eigenvalues $\lambda(x)$ of the Jacobian matrix of the response system satisfies

$$\left\| \frac{\partial f_i}{\partial x_k} + \frac{\partial f_i}{\partial a_i} \frac{\partial a_i}{\partial x_k} - \delta_{ik} \lambda(\mathbf{x}_{ap}) \right\| = 0 \quad (i,k = 1, \ldots, n; i,k \neq j), \tag{3}$$

where $\lambda(\mathbf{x}_{ap})$ is related to the trajectory of the driving system, δ_{ik} is the Kronecker delta function, and $\partial f_i / \partial x_k$ and $(\partial f_i / \partial a_i)(\partial a_i / \partial x_k)$ are evaluated at $\mathbf{x} = \mathbf{x}_{ap}$ and $a_i = \overline{a_i}$. We know that the Lyapunov exponent that represents the character of the system is a long time average quantity along the orbits of the system. If we choose a suitable ϵ_i in Eq. (2) so that all the $\lambda(\mathbf{x}_{ap})$ along the trajectory are negative, the sub-Lyapunov exponents of the response system will be negative. Synchronization can be achieved.

For the case of no drive variable, the response system has the form of Eq. (1). Now we also adjust the parameter a_i of the response system as in Eq. (2); then Eq. (3) becomes

$$\left\| \frac{\partial f_i}{\partial x_k} + \frac{\partial f_i}{\partial a_i} \frac{\partial a_i}{\partial x_k} - \delta_{ik} \lambda(\mathbf{x}_{ap}) \right\| = 0 \quad (i,k = 1, \ldots, n), \tag{4}$$

The question is how we choose a suitable ϵ_i in Eq. (2) so that all the $\lambda(\mathbf{x}_{ap})$ of Eq. (4) along the orbits become negative.

The problem of determining the roots $\lambda(\mathbf{x}_{ap})$ of Eq. (3) when $n \geqslant 4$ and the roots $\lambda(\mathbf{x}_{ap})$ of Eq. (4) when $n \geqslant 3$ can become tedious at best. Fortunately, what is required is not these roots, but simply the region of ϵ_i in which all the roots $\lambda(\mathbf{x}_{ap}) < 0$. The answer to this problem is well known and does not require a knowledge of the roots $\lambda(\mathbf{x}_{ap})$. The characteristic equation (3) or (4) is a polynomial equation of order n in λ,

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0 \quad (a_0 = 1). \tag{5}$$

A necessary and sufficient condition for all roots of Eq. (5) to have negative real parts is that

$$a_{2k} > 0, \quad \Delta_{2k+1} > 0 \quad (k = 0, 1, \ldots)$$

or

$$a_{2k+1} > 0, \quad \Delta_{2k} > 0, \tag{6}$$

where $\Delta_i(a_1, \ldots, a_i)$ are the so-called Hurwitz determinants of order i. These conditions are referred to as the Routh-Hurwitz stability criteria. From these criteria we can determine the control coefficient ϵ_i. We give some examples to explain the process of synchronizing in detail.

III. NUMERICAL EXPERIMENTS

A. Rössler model

For the Rössler system

$$\dot{x} = -y - z, \tag{7a}$$

$$\dot{y} = x + ay, \tag{7b}$$

$$\dot{z} = b + z(x - c) \tag{7c}$$

we consider z as the drive variable; then the response system is

$$\dot{x} = -y - z, \quad \dot{y} = x + ay. \tag{8}$$

Let

$$a = \overline{a} - \epsilon (y - y_{ap}), \tag{9}$$

where $\overline{a}$ is the nominal value of the parameter a, y_{ap} is the aperiodic orbit of drive system, and y is the orbit of response system. When synchronization is implemented, $y = y_{ap}$ and $a = \overline{a}$. According to Ref. [1], we set $a = b = 0.2$ and $c = 9.0$. The Jacobian of the response system is

$$\begin{pmatrix} -\lambda & -1 \\ 1 & \overline{a} - y_{ap}\epsilon - \lambda \end{pmatrix} = 0, \tag{10}$$

where λ is the eigenvalue of the Jacobian of the response system. By requiring $\lambda_{1,2} < 0$, one has

$$|\epsilon| > \frac{2 + \overline{a}}{y_{ap}}. \tag{11}$$

To avoid too large ϵ, we let the response system run freely when $|y_{ap}| \leqslant 0.1$, that is,

$$\epsilon = \begin{cases} \dfrac{2 + a + k_1}{y_{ap}} & \text{if } |y_{ap}| > 0.1 \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Here $k_1 > 0$ is the adjusting parameter. In our synchronizing, we set $k_1 = 0.1$; the initial point of the drive system is (5.0, -10.0,0.03) and the initial point of the response system is (0.1,-6.0,0.03). The driving and response systems have been integrated by using a stable fixed-step fourth-order Runge-Kutta method with a step size of $\tau = 0.01$ time units. Figure 1 shows the result of synchronization. To determine the range of the adjusting parameter k_1, we now investigate the larger sub-Lyapunov exponent spectra of the response system. The sub-Lyapunov exponents $\lambda(k)$ are defined by variational equations of the response system

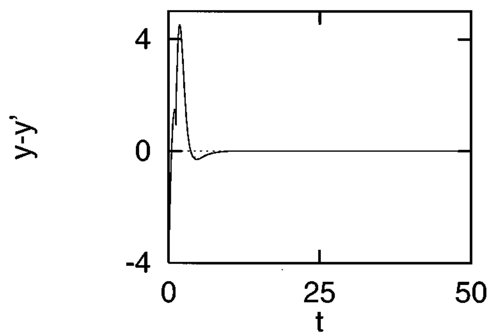

FIG. 1. Synchronization of the Rössler system using z as the drive variable and the subsystem (x,y) as the response system at the parameters $a=b=0.2$ and $c=9.0$. The initial point of the driving system is $(5.0,-10.0,0.03)$ and the initial point of the response system is $(0.1,-6.0,0.03)$.

$$\delta\dot{x}=-\delta y, \quad \delta\dot{y}=\delta x+a\,\delta y, \tag{13}$$

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\ln\frac{\sqrt{\delta^2 x+\delta^2 y}}{\sqrt{\delta^2 x(0)+\delta^2 y(0)}}. \tag{14}$$

Here $\delta x=x-x_{ap}$ and $\delta y=y-y_{ap}$ define the deviations of the response system from the aperiodic orbit, determined by the drive system. The necessary and sufficient condition of synchronization of the two systems is $\lambda<0$. Denoting $\epsilon=k/y_{ap}$ when $|y_{ap}|>0.1$ and $\epsilon=0$ when $|y_{ap}|\leqslant 0.1$, we get the larger sub-Lyapunov spectrum as in Fig. 2. It is negative when k varies from 0.1 to 100, so the synchronization can be implemented in this range. Comparing this range with Eq. (11), we can find that they are different. The range determined by Eq. (11) is contained in that of Eq. (14) which is less than zero. This is because Eq. (11) needs every step of the response system to approach the aperiodic orbit of the drive system, while Eq. (14) is an average effect over long times. It confirms that all the sub-Lyapunov exponents of the range determined by the Routh-Hurwitz criteria are negative.

For the case of no drive variable, we should find the range of adjusting parameter by Eqs. (4) and (6). But in this special example, we can still use Eq. (11) to implement the synchronization for the following reason. Checking the trajectory of the Rössler system in phase space, we find that the variable z is always positive and the variable x varies between -13.0 and 17.0. This means that $x-c$ is negative in most regions of the trajectory. In other words, Eq. (7c) is stable in most regions of the trajectory. So we only need to consider Eqs. (7a) and (7b). Figure 3 shows the result. Obviously, Fig. 3 is similar to Fig. 1.

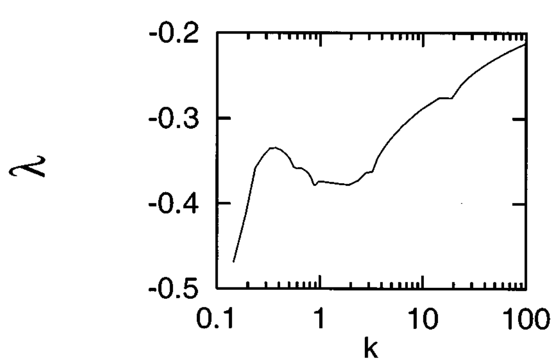

FIG. 2. Larger sub-Lyapunov exponent λ versus weight k of adjusting parameters in the Rössler system. Use z as drive variable.

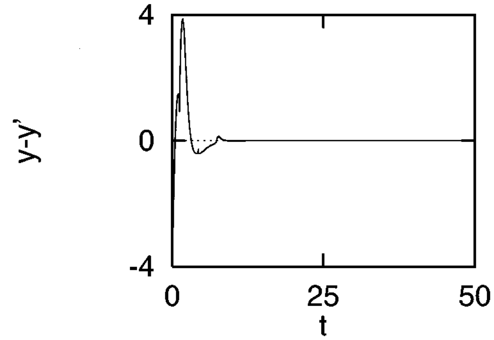

FIG. 3. Synchronization of Rössler system in the case of no drive variable. The initial point is the same as in Fig. 1.

B. Lorenz model

The Lorenz system

$$\dot{x}=\sigma(y-x), \quad \dot{y}=\gamma x-y-xz, \quad \dot{z}=-bz+xy. \tag{15}$$

Let z be the drive variable and (x,y) the response system and adjust the parameter γ as

$$\gamma=\overline{\gamma}-\epsilon(y-y_{ap}), \tag{16}$$

where $\overline{\gamma}$ is the nominal value of the parameter γ and the meanings of y,y_{ap} are the same as in Eq. (9). The Jacobian of the response system is

$$\begin{pmatrix} -\sigma-\lambda & \sigma \\ \gamma-z_{ap} & -1-\epsilon x_{ap}-\lambda \end{pmatrix}=0. \tag{17}$$

With $\lambda_{1,2}<0$ we have

$$\epsilon x_{ap}>\overline{\gamma}-z_{ap}-1. \tag{18}$$

According to Ref. [1], we set $\sigma=10$, $b=8/3$, and $\gamma=60$. Figure 4 is the result when we set $\epsilon=(60-z_{ap})/x_{ap}$ when $|x_{ap}|>0.1$ and $\epsilon=0$ when $|x_{ap}|\leqslant 0.1$. Noting

$$\epsilon=\begin{cases} \dfrac{\overline{\gamma}-z_{ap}-1+k}{x_{ap}} & \text{if } |x_{ap}|>0.1 \\ 0 \quad \text{otherwise,} & \end{cases} \tag{19}$$

we get the larger sub-Lyapunov spectra as in Fig. 5.

For the case of no drive variable, we also adjust the parameter γ as in Eq. (16). Substituting Eq. (16) into Eq. (15), we get its Jacobian matrix

$$J=\begin{pmatrix} -10.0 & 10.0 & 0 \\ 60.0-z_{ap} & -1-\epsilon x_{ap} & -x_{ap} \\ y_{ap} & x_{ap} & -\dfrac{8}{3} \end{pmatrix}. \tag{20}$$

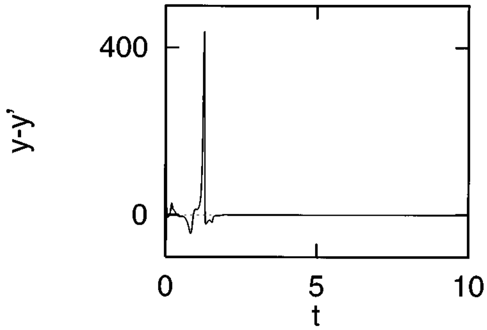

FIG. 4. Synchronization of the Lorenz system using z as the drive variable and subsystem (x,y) as the response system at the parameters $\sigma=10$, $\gamma=60$, and $b=8/3$. The initial point of the driving system is (2.0,20.0,10.0) and the initial point of the response system is (20.0,10.0,30.0).

Making the three eigenvalues of Eq. (20) become negative, from Eqs. (4) and (6) we get

$$\epsilon=\begin{cases} 1 & \text{if } x_{ap}>1 \\ -1 & \text{if } x_{ap}<-1 \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Figure 6 shows the result. Comparing Fig. 6 with Fig. 4, one can see that the effect of synchronization in Figure 6 is better than that in Fig. 4. This means that the case of no drive variable is sometimes better than having a drive variable.

IV. EFFECTS OF NOISE

To test whether this method can be used in experiments, we now study the effect of noise. In this section we consider Gaussian white noise ξ having a zero mean and standard deviation equal to one, generated by using the Box-Müller method [15]. Here we discuss only the case of no drive variable and introduce additive noise in the form

$$x'=x+\rho\xi, \tag{22}$$

where ρ denotes the intensity of external noise and x the variable of the response system. This noise is applied at each Runge-Kutta integration step. Figure 7 shows the result. Figure 7(a) represents the case of the Rössler system with

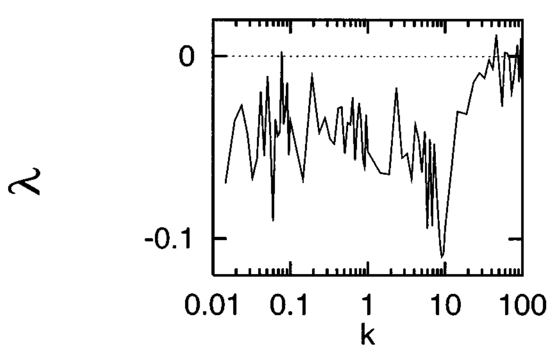

FIG. 5. Larger sub-Lyapunov exponent λ versus weight k of adjusting parameters in the Lorenz model.

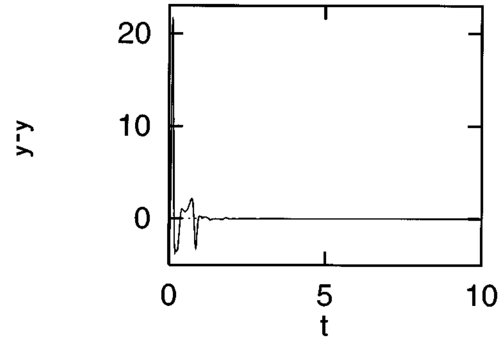

FIG. 6. Synchronization of the Lorenz system in the case of no drive variable. The initial point is the same with Fig. 4.

$\rho=5.0\times10^{-3}$ and Fig. 7(b) the case of the Lorenz system with $\rho=1.0\times10^{-2}$. From this figure one can see that the effect of adding noise is to weaken the synchronization. This effect becomes more and more intense as the level of noise is increased until synchronization is completely lost.

V. CONCLUSION

In conclusion, we have shown that the synchronization of a chaotic system can be implemented by adjusting the parameter of the response system. This method is mainly used in the case in which the synchronization cannot be implemented by the method of Ref. [1]. The operating range of the method can be determined by using the Routh-Hurwitz criteria. The method works if the largest sub-Lyapunov exponent of the response system is negative. To avoid large parameter changes, we let the response system run freely when the required parameter change is large.

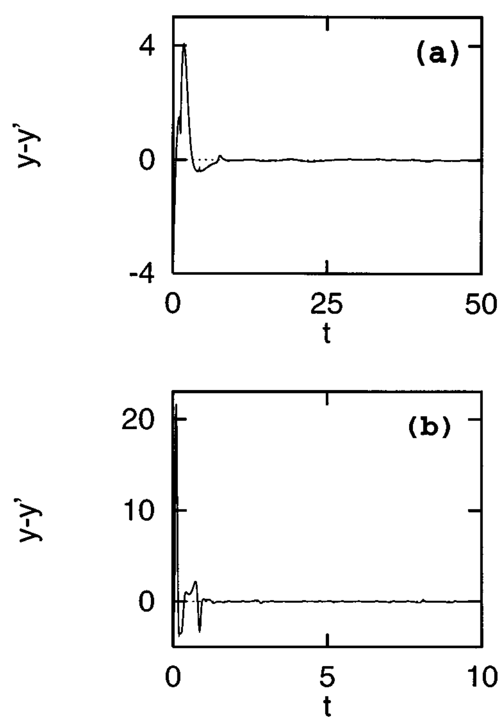

FIG. 7. Synchronization of no drive variable in the case of adding noise. (a) Rössler system with the intensity of noise $\rho=5.0\times10^{-3}$ and (b) Lorenz system with the intensity of noise $\rho=1.0\times10^{-2}$.

We point out in particular that our method can be used in the case where there is more than one positive sub-Lyapunov exponent of the response system. Reference [10] pointed out that in controlling chaos, to stabilize the chaos of higher order, multivariable control has to be used. The minimal number of controlled variables has to be equal to the number of positive Lyapunov exponents of the unperturbed system. From our numerical simulation we know that this is not true in synchronizing a response system. On the other hand, our method is robust against noise.

ACKNOWLEDGMENTS

This work was supported partially by the National Science Foundation of China and the Science Foundation of China Academy of Engineering Physics.

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[2] R. He and P. G. Vaida, Phys. Rev. A **46**, 7387 (1992).
[3] N. F. Rulkov *et al.*, Phys. Rev. E **51**, 980 (1995).
[4] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
[5] J. M. González-Miranda, Phys. Rev. E **53**, 5 (1996).
[6] K. M. Cuomo and A. V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993).
[7] S. Hayes, C. Grebogi, and E. Ott, Phys. Rev. Lett. **70**, 3031 (1993).
[8] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **74**, 5028 (1995).
[9] U. Parlitz, L. Kocarev, T. Stojanovski, and H. Preckel, Phys. Rev. E **53**, 4351 (1996).
[10] K. Pyragas, Phys. Lett. A **181**, 203 (1993).
[11] T. Kapitaniak, Phys. Rev. E **50**, 1642 (1994).
[12] Cheng Yanxiang and Wang Guangrui, Acta Phys. Sin. **44**, 1382 (1995).
[13] Y.-C. Lai and C. Grebogi, Phys. Rev. E **47**, 2357 (1993).
[14] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[15] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, New York, 1986).